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Wave–vortex dynamics

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Abstract. Numerical studies of two-dimensional turbulence show the importance of localised structures, both in the small scales where intermittent transfers occur and in the large scales dominated by coherent vortices. This work is an attempt to establish a more appropriate theoretical framework than the usual Fourier representation. We present a dual representation, mixing continuous field and point vortices. The difficulties arising from the redundancy of the representation are solved through a hypothesis of mutual advection. The resulting equations conserve the natural invariants of the system. As a first application, we show that introducing a weakly charged point vortex in a continuous field with quasiperiodic behaviour induces a purely Lagrangian chaos. As the point vortex grows in intensity, it becomes quasistationary and the induced phase mixing dominates the behaviour of the continuous field.

1. Introduction

There is a growing interest, both in theoretical physics and in fluid dynamics, to study the statistical properties of 2D Navier–Stokes equations at high Reynolds number. Natural motivations lie in the approximate two dimensionality of large scale dynamics of geophysical fluids and rapidly rotating fluids in astrophysics. In addition, some experimental data are available from several laboratory experiments that have been designed in the last few years (Hopfinger *et al* 1982, Sommeria and Verron 1984, Couder 1984). On the other hand, with the present generation of supercomputers it is possible to achieve high resolution simulations at large Reynolds number (Brachet *et al* 1986), whereas this is not yet the case for 3D turbulence, for which the high Reynolds number range is not accessible by direct simulation.

A striking feature of 2D turbulent flows, observed both numerically and experimentally, is their ability to develop long lived vortices which concentrate a large amount of vorticity. These coherent structures obtain for a very large range of conditions (Basdevant *et al* 1981, McWilliams 1984, Hopfinger *et al* 1982) in decaying systems and in forced experiments as well. They bear strong similarities with observed eddies in the ocean (Ring Group 1981). The vortices are clearly seen in figure 1 which shows a vorticity field, taken from the history of a numerical integration of 2D Navier–Stokes equations in a squared periodic box (Benzi *et al* 1987). The high resolution of this experiment (512×512) allows generation of vortices within small scales of motion, not seen in previous studies.

In previous studies (Babiano *et al* 1987, McWilliams 1984, Benzi *et al* 1986), it was suggested by qualitative and quantitative arguments that 2D turbulent flows can

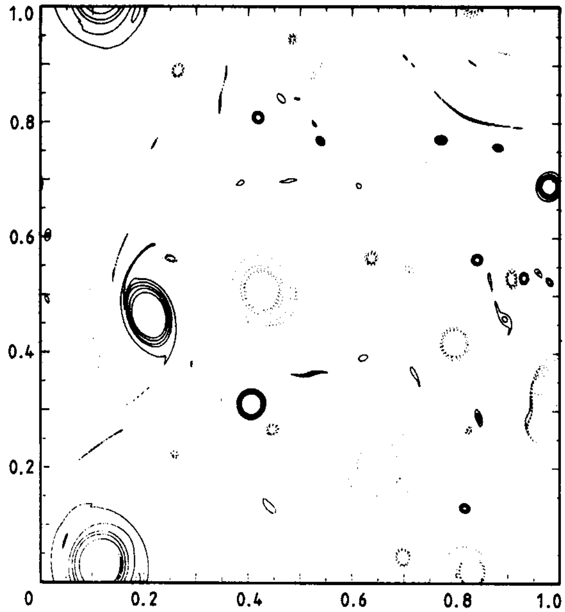


Figure 1. Contour plot of the vorticity field obtained by a numerical simulation for a decaying turbulence (from Benzi *et al* 1987). The model employed is a spectral model with 512×512 grid resolution. Dissipation is prescribed by a superviscosity term of the form $-\nu\Delta^3\Psi$, Ψ being the stream function of the system and $\nu = 2 \times 10^{-9}$. The boundary conditions are periodic with period 2π.

be somehow split into two main components, coherent structures on one side and background field on the other, which exhibit two kinds of dynamical behaviour. Coherent structures are characterised by a strong correlation between the vorticity field and the stream function, which implies weak non-linearities and inhibition of the enstrophy cascade towards small scales (McWilliams 1984, Benzi *et al* 1986). On the other hand, the advection by coherent structures plays the major role in the evolution of the background vorticity field (Babiano *et al* 1987). An enstrophy cascade towards small scales is observed within the last region, leading to a k^{-3} partial spectrum in agreement with the classical phenomenology (Babiano *et al* 1987, Benzi *et al* 1986). Within the background field, the vorticity behaves essentially as a passive scalar.

Although the flow is usually observed from an Eulerian point of view, the turbulent behaviour is partly of Lagrangian source. By Lagrangian turbulence, we mean the chaotic and irregular trajectory of a passive particle advected by the flow. It has been shown (Aref 1984) that Lagrangian turbulence is present in two dimensions even if the velocity field behaves periodically.

A more delicate question concerns the stability and the dynamics of coherent vortices. Many authors have observed that these structures are close to local inviscid solutions of the Euler equation. It is clear that any radial distribution of vorticity within a disc is such a solution. Why some profiles are robust to dissipation and external perturbations still remains unclear. A particular solution was given by Leith (1984) which minimises the enstrophy for a given energy or total vorticity. A class of non-axisymmetric solutions, with two vortices of opposite signs pairing in a compact structure, is known as modons in the literature (Flierl *et al* 1980). In most cases, observed vortices persist without significant change in their shape for a very long time.

Collisions between vortices develop transient zones of intense straining in the background field. When two vortices with the same sign get sufficiently close to each other, the interaction is strongly inelastic and can lead, in some rare events, to the absorption of the weaker vortex by its companion.

In any case, many Fourier modes are needed to correctly simulate the dynamics of coherent structures. The non-local description provided by the spectral expansion is not well suited to capture the characteristics of the vortices in a few numbers. Another possible way of tackling the problem is to study the dynamics of point vortices (Chorin 1973). It is *a priori* clear that coherent structures can be represented by point vortices with the same degree of difficulty as Fourier modes. However, at some distance from a localised vorticity concentration, the induced velocity is close to the velocity induced by a point vortex located at the centre of mass of the concentration with the same total vorticity. Provided that a scale separation exists between the lifetime of the vortex and the turnover time of the incoherent part of the field, it is tempting to replace the continuous vortex by a single point vortex. An appealing feature of point-vortex systems is the Hamiltonian flow of their dynamics. A major consequence is the existence, for a large but finite number of vortices, of statistical equilibrium properties which have been investigated by many authors (Onsager 1949, Novikov 1975, Frohlich and Ruelle 1982).

A second class of motivations arises from the dynamics of small scales. In 2D turbulent flows, the enstrophy is cascaded towards the small scales through non-linear interactions (Kraichnan 1967, Rose and Sulem 1978) but the measure of the spatial domain in which such transfers are active decreases as the scale decreases (Basdevant and Sadourny 1983, Benzi *et al* 1984). In the absence of any organised large scale flow, intermittency is able to steepen the energy spectrum slope (Basdevant *et al* 1981, Benzi *et al* 1984). Point vortices are again a good candidate to provide a local description of small scale activity. Dissipation can be achieved by adding random noise to the deterministic motion and feedback effects to the large scales can still be obtained by clustering of the vortices.

All the above considerations suggest that the duality local-global or Lagrangian-Eulerian can be studied by a combined approach where point vortices and Fourier modes are considered for the 2D Euler equations.

One may wonder if a redundant description of any physical system can be of interest to obtain a better understanding of the dynamics. Our approach is motivated by the physical assumption that 2D Euler equations seem to support this duality and that a good understanding of the physical properties can be achieved by explicitly considering global and local features in a coupled model. We are aware that the redundancy of this description will induce mathematical problems. The functional space introduced in this way will show, for instance, lack of orthonormality. As a consequence we may expect that some of the invariants of motion are missed.

In this paper, we discuss these difficulties as well as the consequences of this approach. The motivation of this study is not to solve all the problems but to encourage future works in a direction which we believe promising for a better understanding of 2D turbulence. In § 2, we discuss a possible model for wave-vortex interactions which fulfils prescribed physical requirements. We start discussing our problem in the continuous limit and then we define, in § 3, the projection operator on a truncated set of Fourier modes. The ambiguities of such projections are discussed in terms of the physical constraints. Section 4 is devoted to a discussion of a minimal model based on two Fourier modes and one point vortex. Although the model cannot be solved analytically we derive some simple consequences which can be related to recent numerical simulation of 2D turbulent flows.

In § 5, we study by numerical integration the possible models proposed in § 2. Five Fourier modes are considered together with one point vortex. We show that Lagrangian turbulence is generated by the motion of the vortex. A striking feature of these simulations is that the conservation of total energy does not seem to be an essential constraint of the dynamics although the enstrophy conservation does. We conclude in § 6 with a list of addressable problems which can be possibly tackled by the method proposed in this paper. Most of them are relevant both for 2D turbulence theory and for geophysical fluid dynamics.

2. Formal properties

We consider a two-dimensional rotational flow in a domain D with rigid or periodic boundary conditions in which local concentrations of vorticity exist in subdomains d_i. Inside a subdomain d_i, the vorticity is much larger than in the surrounding fluid. We assume that the dynamics preserve these concentrations for a long time. The previous section recalls that numerical experiments and observations of natural systems strongly support this hypothesis.

The fundamental simplification of our approach is to replace each patch of concentrated vorticity by an isolated vortex with a vorticity charge equal to the integrated vorticity over the patch. The vorticity field is then separated into two components, a continuous field and a discrete distribution of vortices.

The self-interaction of the continuous field leads to the well known partial differential vorticity equation

$$\mathrm{D}q/\mathrm{D}t = \partial q/\partial t + J(\psi, q) = F \tag{1}$$

where ψ is the stream function such that $q = \Delta\psi$. J is the Jacobian operator and F stands for forcing and dissipation.

If F is set to zero, with appropriate boundary conditions, equation (1) conserves the energy $E = \int_D \text{del}\, \psi^2\, \mathrm{d}\sigma$ and all moments of the vorticity q, among which the enstrophy $Z = \int_D \Delta\psi^2\, \mathrm{d}\sigma$. This latter quantity plays a special role in the classical phenomenology of 2D turbulence (Kraichnan 1967, Rose and Sulem 1978). When the system is forced into a narrow band of wavenumbers, the injected enstrophy is transferred by non-linear interactions towards small scales, where it is dissipated. Unlike 3D turbulence, the energy is not cascaded towards small scales but is essentially transferred towards large scales where it can build up coherent structures.

In the inviscid case, a system of point vortices is an exact solution of the Euler equation. Each vortex conserves its vorticity during the evolution. An important feature of this system is the existence of a Hamiltonian from which the equations of motion are derived (Kraichnan and Montgomery 1980).

In this formulation, the stream function of an individual vortex of vorticity q located in r_0 is given by the Green function $G(x, r)$ of the Poisson equation in D:

$$\Delta G(x, r_0) = \delta(x - r_0) \tag{2}$$

$$\psi_0(x) = \Gamma G(x, r_0). \tag{3}$$

The self-energy and self-enstrophy of this vortex are infinite, but the interaction energy or Hamiltonian of a family $\{\Gamma_i, r_i\}$ of vortices is finite:

$$H = E_{\text{int}} = -\tfrac{1}{2} \sum_{i \neq j} \Gamma_i \Gamma_j G(r_i, r_j) \tag{4}$$

where H is an integral of motion and the motion of each vortex is determined by

$$\dot{r}_i = \sum_{j \neq i} \Gamma_j k \times \text{grad } G(r_i, r_j) = k \times \text{grad } \psi_i(r_i). \tag{5}$$

Here ψ_i is the stream function induced by the system of vortices except the vortex i.

2.1. *Equations of motion*

As stated above, we expand the vorticity field in a discrete distribution of vortices $q_v(x) = \Sigma_i \Gamma_i \delta(x - r_i)$ and a continuous field $q_w(x)$. If the point vortices are considered as limits for local concentrations of vorticity, the latter representation is redundant. This mathematical ambiguity must be removed by a physical ansatz in order to obtain a closed set of equations of motion.

We make the basic assumption that the intensity of the vortices is unchanged through the interaction with the zonal flow. The nature of this assumption is heuristic: there is clear evidence from experiments which produce localised eddies that these latter vary in intensity following the interaction with their environment. However, we want precisely to eliminate the complex mechanisms which generate and maintain the eddies in order to study at first the mutual effects of eddies and continuous field on their respective dynamics. On the other hand, a variation of the amplitude of a vortex implies a local associated variation of the vorticity of the mean field. Since we intend to apply this formalism with a truncated expansion of the continuous field over large scales only (see § 3 below), we can hardly achieve local exchanges within this framework.

The sole variable quantity associated with the vortices is their position. Each vortex is advected by its companions and by the continuous field. For the ith vortex, we have

$$\dot{r}_i = k \times \text{grad}(\psi_w + \psi_i)\big|_{r=r_i}. \tag{6}$$

On the other hand, the continuous vorticity is advected by the continuous field and by the vortices. We have

$$\frac{\partial}{\partial t} q_w + J(\psi_v + \psi_w, q_w) = F \tag{7}$$

where ψ_v is the total stream function of the system of vortices. In the following, φ_{vi} denotes the normalised stream function of the ith vortex centred in $r = r_i$. So we have $\psi_v = \psi_i + q_i \varphi_{vi}$.

Apparently, equation (7) is not defined at the location of vortices and may contain spurious contributions from their vicinity. We can see that this is not the case by considering a vortex with unit vorticity centred in 0 whose stream function in the local cylindrical coordinate system (ρ, θ) is $\varphi_v(\rho) = \ln(\rho/\rho_0)$. The Jacobian of φ_v and q_w is then

$$J(\varphi_v, q_w) = \frac{1}{\rho}\left(\cos\theta \frac{\partial q_w}{\partial y} - \sin\theta \frac{\partial q_w}{\partial x}\right).$$

Since q_w is smooth, we may expand its derivatives in the vicinity of the origin:

$$J(\varphi_v, q_w) = \frac{1}{\rho}\left(\cos\theta \frac{\partial q_w}{\partial y}(0) - \sin\theta \frac{\partial q_w}{\partial x}(0)\right)$$
$$+\tfrac{1}{2}\sin 2\theta\left(\frac{\partial^2 q_w}{\partial y^2}(0) - \frac{\partial^2 q_w}{\partial x^2}(0)\right) + \cos 2\theta \frac{\partial^2 q_w}{\partial x\,\partial y}(0) + \mathrm{O}(\rho). \tag{8}$$

Integrating over a small disc δ of radius a centred in 0, we see from (8) that the local contribution of the vortex is at most of order a^3. If we define $J(\varphi_v, q_w)$ at the origin as

$$J(\varphi_v, q_w) = \lim_{a\to 0} \frac{1}{\pi a^2}\int_{\delta-\{0\}} J(\varphi_v, q_w)\,\mathrm{d}\sigma \tag{9}$$

the previous result shows that this limit is zero. We may thus consider that (7) is formally valid at the location of vortices without difficulties, using the ansatz $J(\varphi_i, q_w)(r_i) \equiv 0$.

We want to warn the reader of a conceptual difficulty in using the partial differential equation (7). Let us denote by Λ the smallest scale of variability of the continuous field which corresponds usually to the dissipation scale but might also be a prescribed truncation when only the large scale components are retained. The coupling with a point vortex described by equation (7) breaks this property since it introduces arbitrary small scales in the motion. In order to recover physical consistency, it is necessary to consider equation (7) as valid when averaged over Λ. However, we will first establish in the following the formal invariants of the motion described by equations (6) and (7), before discussing their physical meaning when applying a spatial average.

2.2. Formal invariants of motion

In the inviscid case, i.e. when $F \equiv 0$, the pair of equations (6) and (7) possesses several integrals of motion. Let us first consider the enstrophy of the continuous field $Z_w = \langle q_w^2 \rangle$, where the brackets mean integration over the whole domain D.

If we assume that the boundaries of D are streamlines or are periodic, Z_w is conserved by the time evolution since $\langle J(\psi_v + \psi_w, q_w^2)\rangle \equiv 0$. This property holds equally for any other moment of the continuous vorticity q_w and bears no differences with the case where the continuous field is considered alone.

The self-energy of an individual vortex cannot be defined and there is no enstrophy for the interaction between the vortices. However, an interaction enstrophy between the continuous field and the vortices can be defined as

$$Z_{vw} = \sum_i \Gamma_i q_w(r_i). \tag{10}$$

Using (6), (7) and (9), we show that Z_{vw} is conserved:

$$\dot{Z}_{vw} = \sum_i \Gamma_i\left(\frac{\partial q_w}{\partial t}(r_i) + \dot{r}_i\cdot \mathrm{grad}\; q_w(r_i)\right) = \sum_i \Gamma_i^2 J(\varphi_{vi}, q_w)(r_i) \equiv 0. \tag{11}$$

As previously, this property still holds if we replace q_w or q_i in (8) by any function of q_w or q_i.

The conservation of all the moments of vorticity in the Euler equation is a consequence of the conservation of vorticity for each particle of fluid. The separate conservation of Z_w and Z_{vw} shows that similar properties are maintained in our system.

In the same way, we define a self-energy of the continuous field and an interaction energy of the continuous field and the vortices:

$$E_w = \tfrac{1}{2}\langle(\text{grad }\psi)^2\rangle \tag{12}$$

$$E_{vw} = -\sum_i \Gamma_i \psi_w(r_i). \tag{13}$$

In addition, we need also to consider the interaction energy of the system of vortices

$$E_v = -\tfrac{1}{2}\sum_{i\neq j} \Gamma_i\Gamma_j \varphi_{vi}(r_j). \tag{14}$$

Using (6) and (7) and integrating by parts, we derive the evolution equations for the three energies

$$\dot{E}_w = -\left\langle \psi_w \frac{\partial}{\partial t} q_w \right\rangle = \langle \psi_w J(\psi_v, q_w)\rangle \tag{15}$$

$$\dot{E}_v = \sum_i \frac{\partial E_v}{\partial r_i}\dot{r}_i = \sum_i \Gamma_i J(\psi_v, \psi_w)(r_i) \tag{16}$$

$$\begin{aligned}\dot{E}_{vw} &= -\sum_i \Gamma_i\left(\frac{\partial \psi_w}{\partial t}(r_i) + \dot{r}_i\cdot \text{grad }\psi_w(r_i)\right)\\ &= \sum_i \Gamma_i(\Delta^{-1}J(\psi_v+\psi_w, q_w)(r_i) - J(\psi_v, \psi_w)(r_i))\\ &= \langle \psi_v J(\psi_w, q_w)\rangle - \sum_i \Gamma_i J(\psi_v, \psi_w)(r_i).\end{aligned} \tag{17}$$

Combining equations (15)-(17), we obtain a conservation law for total energy:

$$\dot{E}_w + \dot{E}_{vw} + \dot{E}_v = 0. \tag{18}$$

Equations (15)-(17) describe the exchanges between the continuous field and the system of vortices. In order to extract energy from the vortices, the velocity of the continuous field must point to the right (left) of the velocity induced by the other vortices at the location of a positive (negative) vortex. In order to extract energy from the continuous field, the stream function of the vortices must be anticorrelated with the deformation field of the continuous vorticity. Notice also that there is no direct exchange between E_v and E_w but that the energy must be conveyed through E_{vw}.

3. Truncated system

We now discuss the modifications induced on (6) and (7) when the continuous field is represented by a finite number of degrees of freedom such that the fluctuations of the continuous field occur at scales larger than Λ.

In the present case, it is most convenient to represent the continuous field as a finite expansion in terms of orthogonal functions in the domain D. When D has simple frontiers with periodic or free-slipping conditions, we choose the eigenfunctions $\varphi_k(r)$ of the Laplacian as a basic set. They verify

$$\Delta\varphi_k = -k^2\varphi_k \tag{19}$$

where k is a wavenumber and $k^2 = |k|^2$.

The allowed wavenumbers are limited to a truncation domain K of the phase space, so that the stream function or any scalar field is

$$\psi_w(r) = \sum_{k \in K} \psi_{wk}\varphi_k(r).$$

Since the Jacobian operator in (1) is quadratic, it contains some contributions from wavenumbers outside K which have to be removed for consistency of the truncation through time evolution. In the classical Galerkin approximation, this is performed by projecting the total Jacobian onto each mode ψ_k of K by a scalar product. We thus replace (7) by

$$\dot{q}_{wk} + \langle \varphi_k(r) J(\psi_v + \psi_w, q_w)(r) \rangle = F_k \tag{20}$$

which can also be put into the form

$$\dot{q}_{wk} + \langle \varphi_k(r) J(\psi_w, q_w)(r) \rangle - \sum_i \Gamma_i \Delta^{-1} J(\varphi_k, q_w)(r_i) = F_k. \tag{21}$$

The main interest of the projection (21) is that it minimises the error on the contribution from the continuous field. The truncation of the Jacobian in K is exactly taken into account and the neglected part is orthogonal to all modes of the truncation. Consequently, in the inviscid case and in the absence of vortices, equation (21) conserves the energy and enstrophy of the continuous flow. Subsequent loss of conservation can be due to the time integration scheme. Restoration techniques (equations (27)-(29)) can be extended to constrain the invariant through time stepping, but it is generally experienced that a sufficiently small time step introduces only negligible fluctuations. Although the higher moments of the vorticity are not strictly conserved, it is generally believed that conservation of energy and enstrophy for the non-linear operator is essential for the study of turbulent inertial ranges.

These properties do not extend straightforwardly when vortices are incorporated in the system. In this case, the formal properties of quadratic integrals established in § 2.2 are not satisfied when the Galerkin approximation is applied to the wave field. More precisely, the truncation does not yield a variation of the interaction energy and enstrophy between waves and vortices which is consistent with (11) and (13).

However, the enstrophy of the waves $Z_{vw} = \frac{1}{2}\sum_{k \in K} q_{wk}^2$ is still conserved:

$$\dot{Z}_w = -\sum_{k \in K} q_{wk} \langle \varphi_k J(\psi_v + \psi_w, q_w) \rangle$$
$$= -\langle q_w J(\psi_v + \psi_w, q_w) \rangle \equiv 0. \tag{22}$$

Equation (22) holds because q_w does not possess, by hypothesis, any components on modes external to the truncation domain K, and therefore the truncated form of the scalar product $\langle q_w J \rangle$ is exact. We assume here an exact computation of the truncated part of the Jacobian $J(\psi$ sub $v, q_w)$. Since q_w is truncated inside K, this is possible using a pseudospectral method and an expansion of ψ_v in eigenmodes truncated at $k_{\max}$ where $k_{\max}$ is the largest wavenumber within K. In this case, the mesh of the colocation grid should be three quarters of the one used for wave-wave interactions only. In practice, these criteria can be somewhat relaxed since there is no need to correct errors beyond the precision of the computer. Similarly, we obtain

$$\dot{E}_w = \sum_{k \in K} \psi_{wk} \langle \varphi_k J(\psi_v + \psi_w, q_w) \rangle = -\langle \psi_w J(\psi_v, q_w) \rangle \tag{23}$$

which coincides with equation (15). It is clear that equation (16) is not modified either since the interaction between vortices is left unchanged by the approximation of the wave field.

The interaction enstrophy between waves and vortices is now

$$Z_{vw}=\sum_{k\in K}\sum_i \Gamma_i q_{wk}\varphi_k((r_i)).$$

Using (6), (20) and (9), its time evolution is given by

$$\dot{Z}_{vw}=\sum_i \Gamma_i\left(J(\psi_w+\psi_v,q_w)(r_i)-\sum_{k\in K}\langle\varphi_k J(\psi_w+\psi_v,q_w)\rangle\varphi_k(r_i)\right). \tag{24}$$

The two terms on the right-hand side of (24) do not cancel because of the existence of a non-zero component of the Jacobian J outside the truncation domain K. In other words, the conservation of Z_{vw} requires that equation (7) is exactly satisfied at the location of each vortex. When we apply approximation (20), equation (7) is no longer satisfied at any prescribed point but only as an average over the whole domain.

In a similar way, we obtain an expression for the variations of the interaction energy $E_{vw}=-\Sigma_i\,\Sigma_{k\in K}\,\Gamma_i\psi_{wk}\varphi_k(r_i)$:

$$\dot{E}_{vw}=\sum_i\left(\sum_{k\in K}\varphi_k(r_i)\langle\varphi_k\Delta^{-1}J(\psi_v+\psi_w,q_w)\rangle-J(\psi_v,\psi_w)(r_i)\right). \tag{25}$$

Here the second term on the right-hand side of (25) balances $\dot{E}_v$ as in the continuous case but the first term does not balance $\dot{E}_w$. The time variation of the total energy is

$$\dot{E}_{\mathrm{T}}=-\sum_i\Gamma_i\left(\Delta^{-1}J(\psi_w,q_w)(r_i)-\sum_{k\in K}\varphi_k(r_i)\langle\varphi_k\Delta^{-1}J(\psi_v+\psi_w,q_w)\rangle\right) \tag{26}$$

which generally is different from zero.

Equations (24) and (26) show that the non-conservation is due to the lack of locality of the dynamics of the continuous field and is recovered as Λ goes to zero. This is to be expected in view of our discussion in § 2.1. However, there is a major difference between E_{T} and Z_{vw}. Whereas Z_{vw} depends only on local distribution of the vorticity field, E_{T} is dependent on non-local quantities, as is seen in (26) through the operation Δ^{-1}. Hence the conservation of energy in the system is subjected to much less severely local constraints than Z_{vw}. It is thus interesting to explore the possibility of recovering E_{T}, even for highly truncated systems. The result is readily obtained if we choose a truncated non-linear tendency which minimises the distance in phase space with the Galerkin approximation under the constraints $\dot{Z}_w=\dot{E}_{\mathrm{T}}=0$. This variational problem is easily formulated, introducing three Lagrange multipliers. Using the previous definitions, we see that the constraints are linear in the $\dot{q}_{wk}$:

$$0=\sum_{k\in K} q_{wk}\dot{q}_{wk} \tag{27}$$

$$0=\sum_{k\in K}k^{-2}\dot{q}_{wk}q_{wk}-\sum_{k\in K}\sum_i k^{-2}\Gamma_i\dot{q}_{wk}\varphi_k(r_i). \tag{28}$$

The variational equation reduces to

$$\dot{q}_{wk}+\langle\varphi_k J(\psi_v+\psi_w,q_w)\rangle+\lambda_1 q_{wk}+\lambda_3\left(\sum_{k\in K}k^{-2}q_{wk}-\sum_{k\in K}\sum_i k^{-2}\Gamma_i\varphi_k(r_i)\right)=0 \tag{29}$$

for each $k\in K$.

The elimination of $\dot{q}_{wk}$ in (27) and (28) using (29) leads to a linear system in the Lagrange multipliers which can be solved at each time step. Then (30) gives the non-linear contribution to $\dot{q}_{wk}$ and the full tendency is obtained by the addition of F_k. The resulting set of equations will be referred to in the following as type-1 equations, type-0 equations being the unmodified set of Galerkin equations.

We can generalise the above-mentioned method in order to conserve the quantity Z_{vw}. In this case we add to equations (27) and (28) the constraint

$$0=-\sum_{k\in K}\sum_{i}\dot{q}_{wk}\varphi_k(r_i)\Gamma_i \qquad (30)$$

and equation (29) becomes

$$\dot{q}_{wk}+\langle\varphi_k J(\psi_v+\psi_w,q_w)\rangle+\lambda_1 q_{wk}$$
$$+\lambda_2\sum_{k\in K}\sum_i\varphi_k(r_i)+\lambda_3\left(\sum_{k\in K}k^{-2}q_{wk}-\sum_{k\in K}\sum_i k^{-2}\Gamma_i\varphi_k(r_i)\right)=0. \qquad (29')$$

Equation (29′) will be referred in the following as type 2. Type-2 couplings present a major drawback: in the limit of vanishing vortices we do not recover the standard Galerkin approximation. Indeed, condition (30) imposes exact vorticity conservation at a vortex location regardless of its amplitude. This implies a set of arbitrary constraints when the vortex amplitude vanishes.

As discussed in § 1, there are essentially two physical situations in which one is interested when studying wave–vortex interactions: (i) a small number of waves interacting with many weak vortices and (ii) a small number of strong vortices interacting with many waves. In the second case the non-conservation of Z_{vw} and E_{T} induces only small fluctuations around the average values, with a variance going to zero as the number of waves increases. Hence the simplest coupling, type 0, should be used in this case. On the other hand, the situation is completely different for case (i) where the continuous field is represented by a small number of waves. In this case we can still distinguish between two main possibilities: (1) $\Lambda^2>\langle(\delta r)^2\rangle$ and (2) $\Lambda^2<\langle(\delta r)^2\rangle$, where Λ is the smallest characteristic scale of the wave field and $\langle(\delta r)^2\rangle$ is the average quadratic distance between vortices, namely

$$\langle(\delta r)^2\rangle=(1/N^2)\Sigma_{i,j}(r_i-r_j)^2$$

N being the number of vortices. Case (1) means physically that the vortices are clustered on a scale smaller than the wave field. Hence it is unphysical to prescribe any localised constraints to the large scale flow using the invariant Z_{vw}. On the other hand, the conservation of energy can play a significant role in this case because of the well known property of 2D flows to transfer energy from small to large scales. Hence we can conclude that a type-1 coupling should be used to investigate the properties of the system.

We now consider the case $\Lambda^2<\langle(\delta r)^2\rangle$. In this case the average distance among vortices is larger than the characteristic scale of the system. We are interested in the limit

$$N\to\infty \qquad \max_i|\Gamma_i|\to 0 \qquad \lim_{N\to\infty} N\max_i|\Gamma_i|=C<\infty.$$

Let

$$\chi_i=\left(J(\psi_w+\psi_v,q_w)(r_i)-\sum_{k\in K}\langle\varphi_k J(\psi_w+\psi_v,q_w)\rangle\varphi_k(r_i)\right).$$

We can consider $\chi_i(t)$ as a stochastic process of variance σ_i^2. Because $\Lambda^2 > \langle(\delta r)^2\rangle$ we can also assume that

$$\langle \chi_i(t)\chi_j(t)\rangle = \delta_{ij}\sigma_i^2.$$

It follows that

$$\langle(\dot{Z}_{vw})^2\rangle = \Sigma_{ij}\Gamma_i\Gamma_j\langle\chi_i\chi_j\rangle = \Sigma_i(\Gamma_i\sigma_i)^2 \leqslant \frac{C^2}{N^2}\Sigma_i\sigma_i^2 \leqslant \frac{C^2}{N}\max_i \sigma_i^2 \xrightarrow[N\to\infty]{} 0.$$

Similar conclusions hold for the energy conservation. The above argument suggests that the difficulty of conserving the formal invariants Z_{vw} and E_T is essentially of the same kind as in case (ii), i.e. a small number of vortices interacting with many waves. Hence type-0 coupling will be physically sufficient to describe the properties of the system.

In view of this discussion we will not consider type-2 coupling to be relevant in wave-vortex dynamics in any case.

4. A simple analytical example

The simplest model that can be discussed is the dynamics of a single Fourier mode interacting with one point vortex. We shall consider a channel geometry $0 < x < L$, $0 < y < D$ with periodic boundary conditions in x and $\Psi_x = 0$ at $y = 0$, D. The wave field Ψ_w is represented by

$$\Psi_w = 2\psi_{snm} \sin nky \sin mrkx + 2\psi_{cnm} \sin nky \cos mrkx \tag{31}$$

where $k = \pi/D$ and $r = D/L$. Without loss of generality we choose $n = m = 1$. Using the definition

$$\psi_{c11} + \mathrm{i}\psi_{s11} \equiv \rho\, \mathrm{e}^{\mathrm{i}\theta} \tag{32}$$

we can easily derive the equations for ρ, θ, ξ and η where ξ and η are the coordinates of the point vortex in the channel.

In this section we consider type-0 dynamics only, type-1 dynamics giving trivial results in this case, namely no time variations. After some simple algebraic computations we obtain

$$\dot{\rho} = 0 \tag{33}$$

$$\dot{\theta} = -rk^2\Gamma \sin 2k\eta \tag{34}$$

$$\dot{\xi} = 2k\rho \cos k\eta \cos(\theta - rk\xi) \tag{35}$$

$$\dot{\eta} = 2rk\rho \sin k\eta \sin(\theta - rk\xi). \tag{36}$$

Let us introduce the variables

$$q = \theta - rk\xi + k\eta \tag{37}$$

$$p = \theta - rk\xi - k\eta. \tag{38}$$

We can rewrite equations for ρ, θ, ξ and η in the following way:

$$\dot{q} = -rk^2\Gamma \sin(q-p) - 2rk^2\rho \cos q \tag{39}$$

$$\dot{p} = -rk^2\Gamma \sin(q-p) - 2rk^2\rho \cos p. \tag{40}$$

These equations show that the motion can be periodic or quasiperiodic which implies that θ, ξ and η are periodic or quasiperiodic functions in time. Thus, while the amplitude of the wave is not affected by the presence of the vortex, the phase of the wave is strongly dependent upon the position of the vortex.

This result suggests that when more than one vortex is present in the system, then the Fourier modes may display random phase behaviour. Recently, Babiano *et al* (1987) have integrated numerically the 2D Navier–Stokes equations randomising the phase of each Fourier mode every time step. At variance with previous results, they found k^{-3} inertial range for the system. Moreover, it is known (Babiano *et al* 1987, Benzi *et al* 1987) that the turbulence field outside coherent structures has a k^{-3} energy spectrum. Therefore we can speculate whether the Kolmogorov-like background field in 2D turbulent flows can be induced by the randomisation of the phases by a mechanism similar to that discussed in this and in the following section.

5. Numerical simulation of wave–vortex interaction

We consider a highly truncated continuous field represented by five Fourier modes which interact with one point vortex. The geometry of the physical system is a channel ($0<x<L$, $0<y<D$) with periodic boundary conditions in x and slip boundary conditions in y. The five Fourier modes we consider are

$$\psi_A = \sqrt{2}\sin 2ky \tag{41}$$

$$\psi_{1c} = 2\sin ky \cos rkx \qquad \psi_{1s} = 2\sin ky \sin rkx \tag{42}$$

$$\psi_{3c} = 2\sin 3ky \cos rkx \qquad \psi_{3s} = 2\sin 3ky \sin rkx \tag{43}$$

where $k=\pi/D$, $r=nD/L$ and n is an integer. The continuous field is given by

$$\Psi_w = \Psi_A\psi_A + \Psi_{1s}\psi_{1s} + \Psi_{1c}\psi_{1c} + \Psi_{3s}\psi_{3s} + \Psi_{3c}\psi_{3c}. \tag{44}$$

Having in mind future geophysical applications, we choose $r=1.2$ and $k=0.5236$, which are characteristic values for an atmospheric channel at mid-latitudes ($L=30\,000$ km, $D=6000$ km and $n=6$). All the numerical simulations were done with a point vortex initially located at $(0, D/2)$. The initial value of the Fourier modes has been chosen arbitrarily once for all the numerical integrations.

We denote by Γ the vorticity of the point vortex. We start by considering the simplest case $\Gamma=0$, i.e. no wave–vortex interaction is acting on the system. The equations of motion for this case can be obtained by projecting the Euler equation onto the subspace spanned by the Fourier modes. It is easy to show that the behaviour of the dynamical system so obtained is quasiperiodic: two frequencies characterised the time evolution of each Fourier mode. This can easily be seen by computing, for instance, the power spectrum of Ψ_A as shown in figure 2.

The quasiperiodicity of the stream function implies a quasiperiodicity of the velocity field which can induce Lagrangian turbulence in the system. This is indeed the case. Figure 3 shows the x coordinate of a Lagrangian particle advected by the quasiperiodic flow. The motion of the particle is strongly chaotic in both directions.

We now consider the case $\Gamma\neq 0$. For small values of Γ, the point vortex will be advected as a Lagrangian particle, i.e. performing a chaotic path in the domain. This chaotic behaviour will perturb the wave field because of wave–vortex coupling, i.e. the Lagrangian turbulence will act as a perturbation on the Eulerian component of the

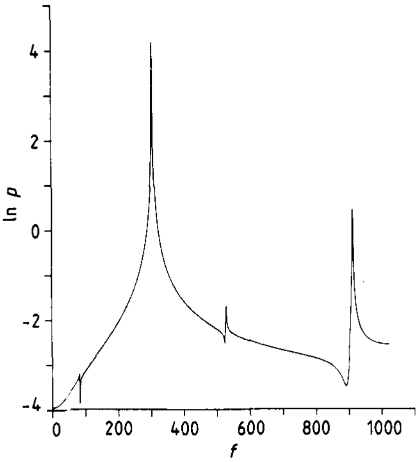


Figure 2. Power spectrum of $\Psi_A(t)$ when no vortex is acting on the Eulerian equations truncated to the five modes (41), (42) and (43). The motion of the system is quasiperiodic with two main frequencies shown in the figure.

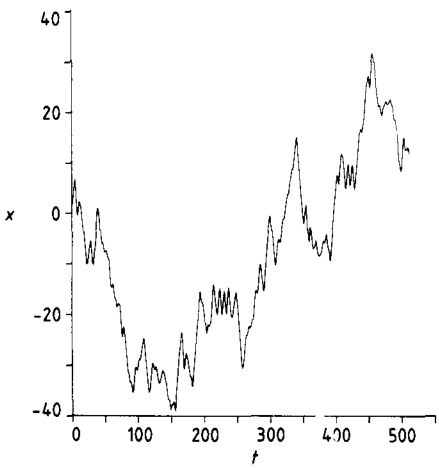
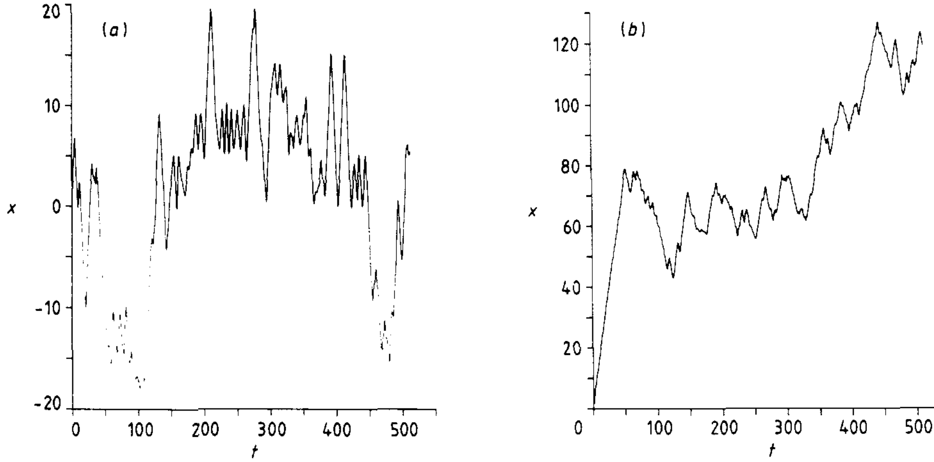


Figure 3. Time behaviour of the x coordinate advected by the quasiperiodic Eulerian flow as described by the five Fourier modes (41), (42) and (43). The Lagrangian turbulence of the particle path is clearly shown by the random behaviour in time of its positions.

flow. The effect of this perturbation depends on the choice of the coupling between the continuous field, i.e. the Fourier modes, and the point vortex. In § 2 we discussed three different kinds of couplings and we observed that type-2 coupling, which forces conservation of Z_{vw}, shows unphysical drawbacks due to its singular behaviour when $\Gamma_i \to 0$. Hence we shall only consider type-0 and type-1 coupling.

Figures 4 and 5 show the x coordinate of the vortex for type-0 and type-1 coupling respectively, and for increasing values of Γ. As expected, for small values of Γ the chaotic behaviour of the vortex path is similar to that of a Lagrangian particle. For large values of Γ the point vortex cannot be considered as a perturbation on the wave field. Figures 4(c) and 5(c) show that the vortex is performing fast and small oscillations



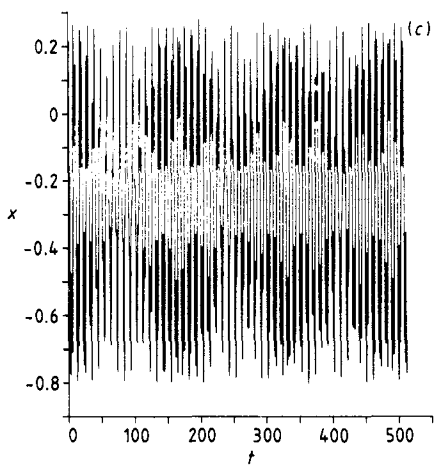


Figure 4. Type-0 wave-vortex coupling. (*a*), (*b*) and (*c*) show the x coordinate of the vortex position respectively for $\Gamma = 10^{-4}$, 10^{-2}, 10^{-1}, Γ being the vortex intensity.

around a fixed position. There is no qualitative difference between type-0 and type-1 coupling. The same is indeed true for all other variables of the dynamical system.

The 'random' perturbation of the Lagrangian turbulence on the Eulerian variables can be easily described in terms of the power spectrum of the Ψ_A variable as previously done for the case $\Gamma = 0$ (see figure 2). Figure 6 shows the power spectrum of Ψ_A for type-0 coupling. The major peaks of the power spectrum are still present for $\Gamma = 10^{-4}$, both for type-0 and type-2 cases, but disappear for $\Gamma = 10^{-2}$. In this case the Lagrangian turbulence 'randomises' completely the Eulerian field. For higher values of Γ the power spectrum of Ψ_A shows some major peaks, i.e. we recover a more ordered behaviour which is probably due to the fixed position of the vortex.

The 'randomisation' of the Eulerian field can be better evidenced by projecting the motion in the phase space of the system onto the two-dimensional subspace spanned by the variables Ψ_{1c}, Ψ_{1s}. The case $\Gamma = 0$ is shown in figure 7. Figure 8 shows the same projection for $\Gamma \neq 0$. As already observed for the power spectrum of Ψ_A, for $\Gamma = 10^{-2}$ a complete 'randomisation' of the Eulerian field can be observed. Figure 8 suggests that the Eulerian field can be approximately described by a 'random phase approximation' of the same kind as that observed for the background field outside the

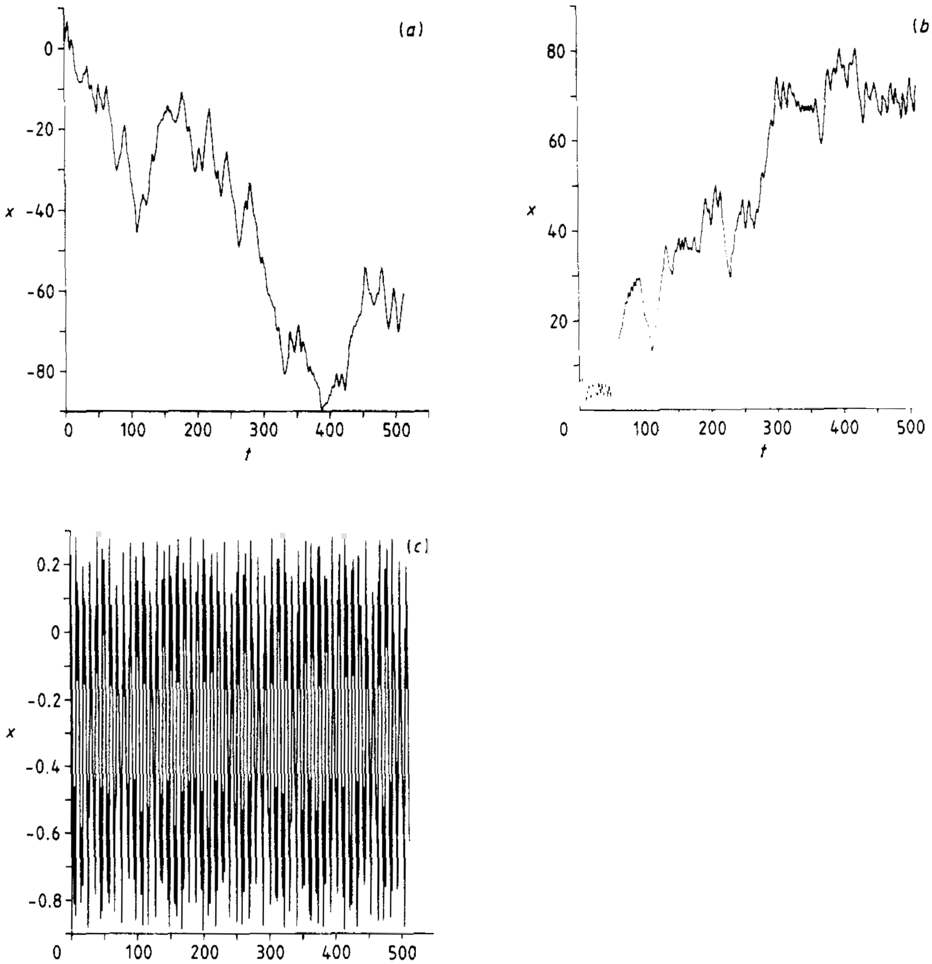


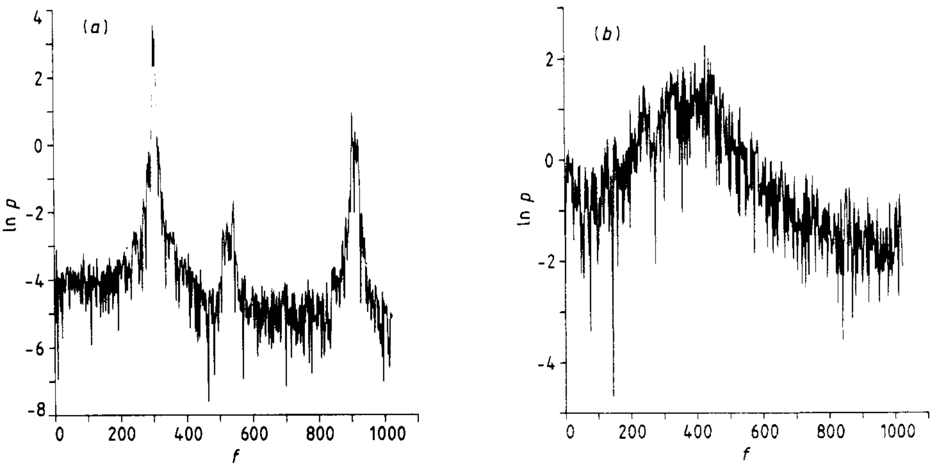
Figure As in figure for type- coupling.

main vortices in numerical experiments of two-dimensional turbulence. Whether or not the two phenomena are connected is still a question to be investigated.

As we have seen, no major differences are observed between type-0 and type-1 coupling. Hence, energy conservation is not a major constraint in the dynamics of the system. This result allows the possibility of inserting point vortex dynamics in numerical spectral codes for high resolution two-dimensional simulations with a relatively modest computational effort. Indeed, for type-0 coupling this possibility can be easily achieved, the same being much more computationally expensive for the type-1 case.

6. Conclusion

In this paper we have shown that a mixed formulation, local-global or discrete-continuous, of two-dimensional Euler equations can be given using few physical requirements.



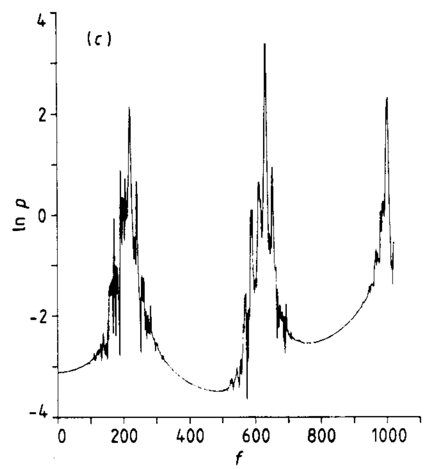


Figure 6. Type-0 wave-vortex coupling. Power spectrum of Ψ_A. (*a*), (*b*) and (*c*) refer respectively to vortex intensity Γ equal to 10^{-4}, 10^{-2} and 10^{-1}.

We point out here several problems for which the proposed method may provide an interesting physical framework.

It is quite possible to generalise the method discussed in § 2 for dissipative flows. There are, in principle, two physical situations: either the vortex field represents small scale dissipative effects or it represents large scale coherent structures like those visible in figure 1. In the first case, the vortices will induce small random perturbations via Lagrangian turbulence and may considerably modify the statistical properties of large scale continuous flows. This will be interesting when multiple regimes are present within the large scale dynamics of the flow since transition properties are expected to depend on the turbulent activity. Examples of this kind can be given for atmospheric dynamics of ultralong planetary waves (Reinhold and Pierrehumbert 1982). Alternatively, when the vortex field represents coherent structures one can discuss the stability of vortex clustering induced by the continuous background field. One particularly interesting question which can be addressed is whether the clustering of vortices, previously observed in numerical simulations (Aref and Siggia 1980), is either emphasised or depressed by the effect of the wave field. Moreover, one can study the

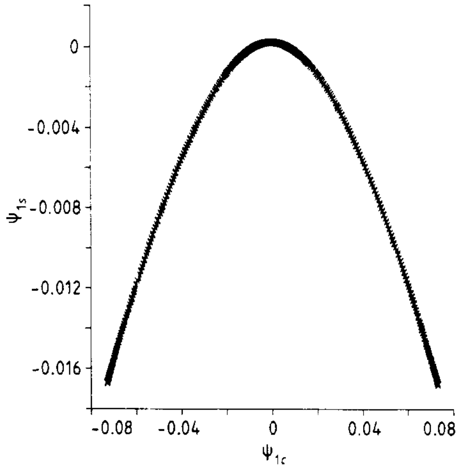


Figure 7. Ψ_{1s} plotted against Ψ_{1c} for the Eulerian flow described by the five Fourier modes (41), (42) and (43). No point vortex is acting on the system.

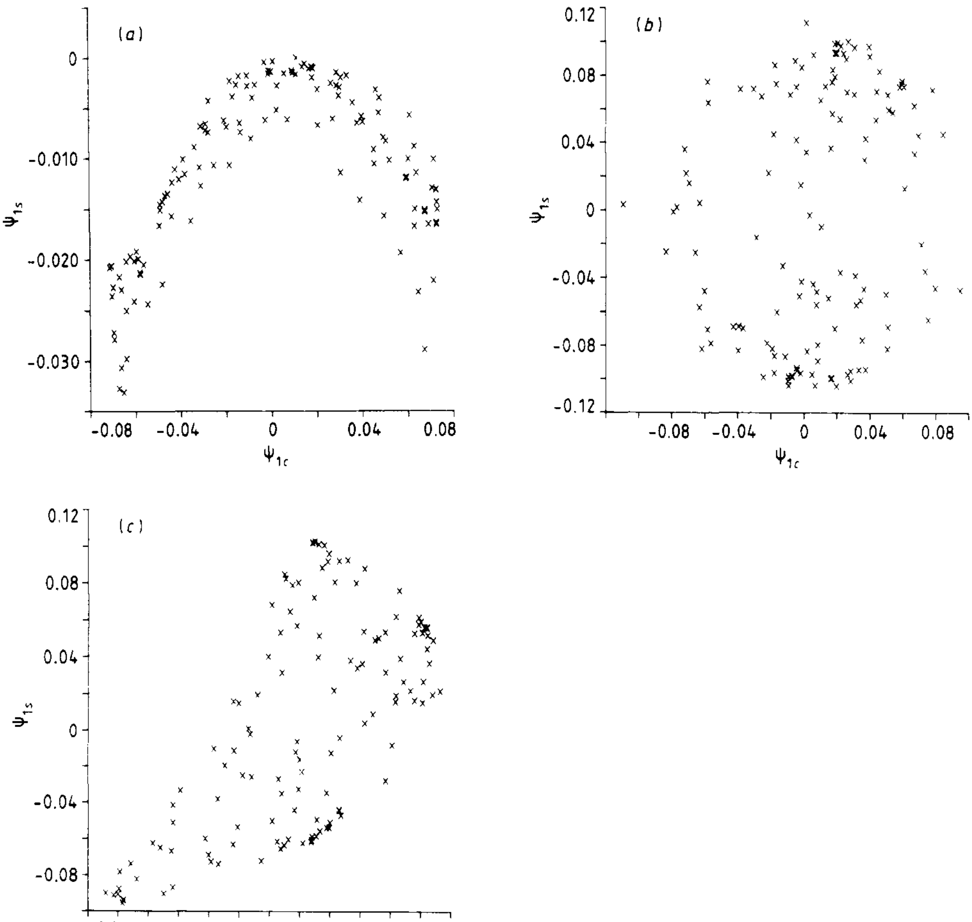


Figure 8. As in figure 7 for type-0 coupling. (*a*), (*b*) and (*c*) refer to $\Gamma = 10^{-4}$, 10^{-2} and 10^{-1}, respectively.

splitting in predictability with respect to amplitude errors in the continuous field and errors in the positions of the vortices. Indeed it has been observed (Basdevant and Legras 1986) that the error field of the first Lyapunov exponent for two-dimensional Navier-Stokes equations is mostly concentrated near the coherent structures. This observation can be studied in detail using one of the proposed methods of § 3.

Such numerical simulations may also provide a way to separate the Lagrangian and Eulerian parts of the turbulence. In both cases which have been mentioned, we may develop an averaging procedure for the fast components of the system: either the waves or the vortices, depending on the limit which is considered. For instance, we would expect that if the vortex field represents coherent structures, the background field would behave essentially as a passive scalar and would satisfy a non-intermittent Kolmogorov law. These questions are presently under investigation by the authors.

Appendix. Stream function of a single vortex

The stream function of a single vortex is expressible in terms of usual functions only in the simplest cases. With periodic boundary conditions in planar geometry, infinite series are obtained which converge slowly in their original form. However, an efficient procedure due to Nijboer and De Wette (1957) leads to a new expansion which converges rapidly.

Let us first recall that, in the infinite plane, a single vortex of charge Γ located at the origin induces a stream function field

$$\psi(r)=\frac{\Gamma}{2\pi}\ln\frac{r}{r_0} \tag{A1}$$

where r is the Euclidian distance to the origin and r_0 an arbitrary constant radius.

On a sphere, a similar expression obtains. For a vortex located at the pole, the stream function is

$$\psi(M)=\frac{\Gamma}{4\pi}\ln(1-\cos\theta) \tag{A2}$$

where θ is the colatitude of the point M.

On a periodic plane, with periods L and D in the x and y directions, such simple expressions are no longer available. For a single vortex located in (x_0, y_0), one has to sum the contributions of all its images located at (x_0+nL, y_0+mD). Indeed, the total vorticity for this distribution is infinite and the stream function diverges at any point. It is thus necessary to add a uniform vorticity field which compensates exactly the charge of the vortex on the mesh area $L\times D$.

Then the vorticity field can be formally expanded as

$$\begin{aligned} q(r;r_0)&=\frac{\Gamma}{LD}\left(\sum_{n,m}\delta(x-x_0-nL, y-y_0-mD)-1\right)\\ &=\frac{\Gamma}{LD}\sum_{n,m}{}'\exp[ik(r-r_0)] \end{aligned} \tag{A3}$$

where $k=2\pi(n/L, m/D)$ is a wavevector and the prime denotes summing over all couples of integers omitting $n=m=0$.

The stream function induced by (A3) is

$$\psi(r;\, r_0) = -\frac{\Gamma}{LD} \sum_{n,m}{}' \frac{1}{k^2} \exp[ik(r-r_0)]. \tag{A4}$$

Equation (A4) is suitable if we need to perform an expansion of ψ in Fourier modes in order to use a transform method. However, if we want to compute $\psi(r)$ at a given location, we need to use a large number of terms of the series to get reasonable accuracy due to the poor convergence of this latter function.

The computational cost can be cut considerably if we use a resummation technique, developed by Nijboer and De Wette (1957) and applied by Seyler (1976) to equation (A4) in a squared geometry. We refer to these authors for a detailed derivation and only give here the resulting expression for $\psi(r;\, r_0)$, which is

$$\psi(r;\, r_0) = -\frac{\Gamma}{LD} \sum_{n,m}{}' \frac{1}{k^2} \exp[ik(r-r_0)] \exp(-k^2L^2/4\pi) + \frac{\Gamma}{4\pi}\left(\frac{L}{D} - \sum_{n,m} E_1\left\{\pi\left[\left(\frac{x-x_0}{L}-n\right)^2 + \left(\frac{y-y_0}{L} - m\frac{D}{L}\right)^2\right]\right\}\right) \tag{A5}$$

where $E_1(u)$ is the exponential integral

$$E_1(u) = \int_u^\infty e^{-t} t^{-1}\, dt.$$

We see that the first term on the right-hand side is now a rapidly converging series. The second term can also be estimated with a few terms of the series since $E_1(u)$ decreases with u more rapidly than an exponential. Furthermore, we do not really need to compute E_1 since only the derivatives of ψ are of interest in the advection. Therefore we obtain

$$\frac{\partial}{\partial x}\psi(r;\, r_0) = \frac{\Gamma}{LD} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty}{}' \frac{4\pi}{k^2}\frac{n}{L} \sin[k(r-r_0)] \exp(-k^2L^2/4\pi) + \frac{\Gamma}{2L} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left(\frac{x-x_0}{L} - n\right) \frac{e^{-u_{n,m}}}{u_{n,m}} \tag{A6}$$

$$\frac{\partial}{\partial y}\psi(r;\, r_0) = \frac{\Gamma}{LD} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty}{}' \frac{4\pi}{k^2}\frac{m}{L} \sin[k(r-r_0)] \exp(-k^2L^2/4\pi) + \frac{\Gamma}{2L} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left(\frac{y-y_0}{L} - m\frac{D}{L}\right) \frac{e^{-u_{n,m}}}{u_{n,m}} \tag{A7}$$

with

$$u_{n,m} = \pi\left[\left(\frac{x-x_0}{L} - n\right)^2 + \left(\frac{y-y_0}{L} - m\frac{D}{L}\right)^2\right].$$

One can easily check that the diverging term for small $r - r_0$ in (A5) leads asymptotically to the form (A1), i.e. in the immediate vicinity of the vortex the boundary effects are not felt.

In the case of channel geometry with periodic conditions with period L in the x direction and a width D with free-slipping conditions in the y direction, we have to consider for an individual vortex all its images produced by mirror symmetry with respect to the rigid boundaries. Thus, we have to sum the contributions of two periodic arrays of vortices of periods L and $2D$ (equation (A1)) in x and y given by formulae (A5)-(A7). The first one has vortices identical to the original vortex at locations $(x_0+nL,\ y_0+2mD)$, and the second one has vortices of opposite sign at locations $(x_0+nL,\ -y_0+2mD)$.

An additional feature of channel geometry is that the total vorticity is zero in a mesh cell $(L, 2D)$ without any need to add a continuous vorticity as in the doubly periodic plane. This implies the existence of an analytic expression of the stream function in terms of complex elliptic functions. The corresponding formulae are given by Morse and Feshbach (1953, p 1239).

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